Superfield equations of motion

1977 J. Phys. A: Math. Gen. 10 2021

(http://iopscience.iop.org/0305-4470/10/11/027)

J. Phys. A: Math. Gen., Vol. 10, No. 11, 1977. Printed in Great Britain. © 1977

Superfield equations of motion

V I Ogievetsky and E Sokatchev

Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, USSR

Received 2 February 1977

Abstract. A general method for deriving the superfield equations of motion is proposed. These equations contain the supplementary conditions of irreducibility with respect to the supersymmetry group. The method uses projection operators which single out the irreducible representations and especially the algebraic roots of these operators. It is found that the standard equations of motion for spin-vector and symmetric tensor fields can in fact be obtained by extracting the square roots of the projection operators for spin-$\frac{3}{2}$ and spin-2, respectively. The spinor superfield equation is deduced and discussed in detail.

1. Introduction

The derivation of the field equations of motion has been considered in many papers which use different approaches. Aurilia and Umezawa (1969) and Belinicher (1974) contain references to the literature on this subject.

We shall consider this problem in connection with superfield theory. The superfields (SF) are rather complicated objects, each of them containing many fields of integer and half-integer spins. Therefore the derivation of adequate equations for them is an urgent and non-trivial task. Notice that only the simplest scalar SF—general and chiral—have previously been considered in detail. Their equations of motion were conjectured by some apt unification of the equations of motion for the fields entering into their composition (Ferrara *et al* 1974, Wess 1976). However, at present, some higher SF are also of interest; in particular, the spinor and vector SF. The spinor SF is connected with an attempt to find the general supersymmetric version (Ogievetsky and Sokatchev 1976) of the Yang–Mills theory. In such a theory the spinor SF is the gauge SF. The vector SF generated by the supercurrent of Ferrara and Zumino (1975) is needed in a possible supersymmetric generalisation of gravitational theory.

The starting point in the above models is the derivation of the free equations of motion. Now it becomes impossible to seek for these equations by some form of sorting due to the higher complexity and increasing number of the operator structures. A certain clear algorithm is needed to obtain them. In the present paper such a procedure is proposed, based on the properties of the projection operators selecting irreducible representations. The new feature consists of establishing the role and using the roots of the projection operators (i.e. the squares of these root operators are the projection operators). We also show that the Rarita–Schwinger equations for the spin-vector field and the Pauli–Fierz equations for the symmetric-tensor field do in fact contain square roots of the projection operators. Therefore the approach under consideration also has some pedagogical value in ordinary field theory. The use of the projection properties permits an easy definition of the Green functions.

The paper is planned as follows. We begin with some necessary information concerning the SF theory and in particular we recall the composition of irreducible supermultiplets in the SF with arbitrary spin. Furthermore we formulate a general idea for the derivation of the equations, which is then illustrated by examples of the standard equations for spin-$\frac{3}{2}$ and spin-2 fields. Following this, the equation for the spinor SF is discussed in detail.

2. Preliminaries

We use the following notations: θ_α denote four-component Majorana spinor coordinates; $\frac{1}{2}\{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu} = \mathrm{diag}(+---)$; $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$; $\sigma_{\mu\nu} = \frac{1}{2}\mathrm{i}[\gamma_\mu, \gamma_\nu]$; $\epsilon^{0123} = 1$; $\bar\theta^\beta = (C^{-1})^{\beta\alpha}\theta_\alpha$ where $C = \mathrm{i}\gamma^0\gamma^2$ is the charge conjugation matrix; and $\Box = \partial_\mu\partial^\mu$. The supersymmetry algebra

$$[J_{\mu\nu}, S_\alpha] = -\tfrac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta, \qquad [P_\mu, S_\alpha] = 0, \qquad \{S_\alpha, \bar S^\beta\} = (\gamma_\mu)_\alpha{}^\beta P_\mu \tag{1}$$

is realised on the SF

$$\Phi_i(x, \theta) = A_i(x) + \bar\theta^\alpha \psi_{\alpha i}(x) + \tfrac{1}{4}\bar\theta\theta F_i(x) + \tfrac{1}{4}\bar\theta\gamma_5\theta G_i(x) + \tfrac{1}{4}\bar\theta \mathrm{i}\gamma_\mu\gamma_5\theta A_i^\mu(x) + \tfrac{1}{4}\bar\theta\theta\bar\theta^\alpha\psi_{\alpha i}(x) + \tfrac{1}{32}(\bar\theta\theta)^2 D_i(x). \tag{2}$$

Here i is some external Lorentz index (e.g. the scalar SF $\Phi(x, \theta)$, the spinor SF $\Phi_\alpha(x, \theta)$, the vector SF $\Phi_\mu(x, \theta)$ etc). We say that the SF $\Phi_i(x, \theta)$ has *external spin j* if it obeys the irreducibility conditions for Poincaré spin j with respect to the index i. For example, the SF $\Phi_\mu(x, \theta)$ has external spin-1 if $\partial_\mu\Phi^\mu = 0$ and spin-0 if $\partial_\mu\Phi_\nu = \partial_\nu\Phi_\mu$.

The irreducible representations of algebra (1) (with non-zero mass) are labelled by the eigenvalues of the second Casimir operator (a generalisation of the square of the Pauli–Lubanski vector) (Lichtman 1971, Salam and Strathdee 1974):

$$W^2 = -m^4 Y(Y+1)$$

where Y is an integer or half-integer called the superspin. A representation with superspin Y contains four ordinary (Poincaré) spins J:

$$J = Y - \tfrac{1}{2}, Y, Y, Y + \tfrac{1}{2}. \tag{3}$$

Sokatchev (1975) has shown that the H7W8SF equation (2) realises reducible representations of supersymmetry. Note the remarkable duality—a SF with external spin j contains four irreducible multiplets with superspins Y:

$$Y = j - \tfrac{1}{2}, j, j, j + \tfrac{1}{2}. \tag{4}$$

The projection operators extracting these irreducible representations out of the SF with arbitrary spin j are also calculated and the corresponding supplementary conditions are derived (Sokatchev 1975).

Finally, we must not forget an important operator—the spinor derivative $\mathscr{D}_\alpha$ (Salam and Strathdee 1975). It obeys the same commutation relations (1) as S_α and anticommutes with S_α:

$$\{S_\alpha, \mathscr{D}_\beta\} = 0.$$

For this reason all the operators invariant under the supersymmetry transformations are constructed out of $\mathscr{D}_\alpha$. In particular, the projection operators mentioned above and the equation of motion operators in which we are interested are polynomials in $\mathscr{D}_\alpha$.

3. Equations of motion

It is instructive to start with an analysis of some features of the standard equations for the ordinary fields. In field theory elementary particles are described by fields which are functions of the coordinates $\phi_i(x)$ transforming according to some representations of the Lorentz group (i stands for a set of Lorentz indices). At the same time these fields give representations of the Poincaré group (with P_μ realised as $\mathrm{i}\partial_\mu$). The latter are *reducible* (at least, because the value of P^2 is not fixed). On the other hand it is natural to associate such characteristics as mass and spin of the particles with *irreducible* representations of the Poicaré group, so we have to impose certain conditions on $\phi_i(x)$ that single out the corresponding irreducible part. First of all, we require that the particle momentum p_μ lies on the mass shell

$$P^2\phi_i = m^2\phi_i. \tag{5}$$

Then, depending on the Lorentz index i, the field can describe one or more spins. It is conventionally assumed that one field describes one spin (as a rule, the highest it contains). In this connection supplementary conditions are imposed:

$$R_{ij}\phi_j = 0 \tag{6}$$

where R_{ij} represents a set of differential operators. Equation (6) excludes all the spins except the highest one.

However, certain troubles arise when the Klein–Gordon equation (5) and the supplementary conditions (6) are written down separately. In this case introducing the interaction can lead to contradictions. Therefore it is strongly preferable to write equations (5) and (6) in the form of a single differential equation

$$\pi_{ij}\phi_j = 0. \tag{7}$$

Now equations (5) and (6) are obtained as corollaries of equation (7). For instance, when spin-1 is described by a vector field $a_\mu(x)$ equations (5) and (6) read

$$\Box a_\mu(x) + m^2 a_\mu(x) = 0, \qquad \partial^\mu a_\mu(x) = 0.$$

These two equations are equivalent to the Proca equation

$$\Box a_\mu(x) - \partial_\mu\partial^\nu a_\nu(x) + m^2 a_\mu(x) = 0. \tag{8}$$

There is one more requirement which concerns the order of the operator π_{ij} in equation (7). We assume that π_{ij} is of first order for Fermi fields and of second order for Bose fields.

How are equations of type (7) deduced to satisfy all the requirements outlined above? The answer is prompted by the Proca equation (8). Let us rewrite it in the form

$$-\Box(\Pi^1)_\mu{}^\nu a_\nu = m^2 a_\mu, \tag{9}$$

where

$$(\Pi^1)_{\mu\nu} = \eta_{\mu\nu} - (\partial_\mu\partial_\nu/\Box)$$

is the projection operator for spin-1. Now it is clear that the field $a_\mu(x) = (-\Box/m^2)(\Pi^1)_\mu{}^\nu a_\nu(x)$ obeys the supplementary condition (of type (6)) $\partial^\mu a_\mu = 0$. Then $(\Pi^1)_\mu{}^\nu a_\nu = a_\mu$ and so equation (9) reduces to equation (5).

This example suggests the general idea. Let Π_{ij} be the projection operator which extracts the representation we are dealing with from the field $\phi_i(x)$. Multiplying by $-\Box$ to a power q which is sufficient to cancel the non-locality, we obtain the equation

$$(-\Box)^q \Pi_{ij}\phi_j = (m^2)^q \phi_i. \tag{10}$$

Thus the irreducible representation is singled out. However, the order of equation (10) may be too high. Suppose that, e.g., $q = 2$ and a second-order equation is required. Then we can find (in general, not uniquely) an operator $\pi = \sqrt{(-\Box)^2\Pi}$ defined by

$$\pi_{ij}\pi_{jk} = (-\Box)^2 \Pi_{ik} \tag{11}$$

and so write an equation of the right order, namely:

$$\pi_{ij}\phi_j - m^2\phi_i = 0. \tag{12}$$

Equation (10) follows from equation (12)

$$(-\Box)^2 \Pi_{ij}\phi_j = \pi_{ik}\pi_{kj}\phi_j = \pi_{ik}(m^2\phi_k) = m^4\phi_i.$$

This means that equation (12) selects the same representation.

A classical illustration for this 'root' trick is the Dirac equation. The bispinor field $\psi_\alpha(x)$ describes spin-$\frac{1}{2}$ only, so here the projection operator is unity and equation (10) takes the form of equation (5):

$$-\Box\psi_\alpha(x) = m^2\psi_\alpha(x).$$

We need a first-order equation so we find the $\pi_\alpha{}^\beta$

$$\pi = \sqrt{-\Box} = \mathrm{i}\not\partial$$

and arrive at the common Dirac equation

$$\mathrm{i}\not\partial\psi - m\psi = 0.$$

The derivation of some other known field equations gives non-trivial examples of how to handle the 'root method'. So, spin-$\frac{3}{2}$ is usually described by a spin-vector field $\psi_{\alpha\mu}(x)$. At the same time this field contains two spins-$\frac{1}{2}$. The supplementary conditions excluding these superfluous spins are

$$\partial_\mu\psi^\mu_\alpha = 0, \qquad (\gamma_\mu\psi^\mu)_\alpha = 0.$$

We want to find an equation of motion containing these supplementary conditions. Consider the projection operator taking spin-$\frac{3}{2}$ out of the field $\psi_{\alpha\mu}$

$$\Pi_{\mu\nu,\alpha\beta} = \eta_{\mu\nu}1_{\alpha\beta} - \frac{2}{3}\frac{\partial_\mu\partial_\nu}{\Box}1_{\alpha\beta} - \frac{1}{3}(\gamma_\mu\gamma_\nu)_{\alpha\beta} + \frac{1}{3\Box}[\not\partial(\partial_\mu\gamma_\nu - \partial_\nu\gamma_\mu)]_{\alpha\beta}. \tag{13}$$

Then we write down the localised operator $(-\Box)\Pi_{\mu\nu,\alpha\beta}$. As it includes second-order derivatives and we need a first-order equation ($\psi_{\alpha\mu}$ is a fermion field), we have to extract the square root of $(-\Box)\Pi$. There is a one-parameter set of such roots, and in the equations obtained, a one-parameter change of field variables $\psi_\mu \to \psi_\mu + \beta\gamma_\mu\gamma^\nu\psi_\nu$ can be made. Finally, the restriction that the equations must correspond to Hermitian

Lagrangians leads to the Rarita–Schwinger set of equations

$$(\not\partial - m)\psi_\mu - \alpha(\partial_\mu \gamma^\nu \psi_\nu + \gamma_\mu \partial^\nu \psi_\nu) + \tfrac{1}{2}(3\alpha^2 - 2\alpha + 1)\gamma_\mu \not\partial \gamma^\nu \psi_\nu + (3\alpha^2 - 3\alpha + 1) m \gamma_\mu \gamma^\nu \psi_\nu = 0. \tag{14}$$

Here α is an arbitrary real parameter.

The next example is the symmetric tensor field $h_{\mu\nu}(x)$ describing spin-2 and superfluous spin-0 and spin-1 too. The corresponding supplementary conditions

$$\partial^\mu h_{\mu\nu} = 0, \qquad h^\mu{}_\mu = 0$$

should follow from the equations of motion. The projection operator for spin-2 is ($\bar\eta_{\mu\nu} = \eta_{\mu\nu} - \partial_\mu \partial_\nu / \Box$):

$$\Pi_{\mu\nu,\lambda\rho} = \tfrac{1}{2}\bar\eta_{\mu\lambda}\bar\eta_{\nu\rho} + \tfrac{1}{2}\bar\eta_{\mu\rho}\bar\eta_{\nu\lambda} - \tfrac{1}{3}\bar\eta_{\mu\nu}\bar\eta_{\lambda\rho} \tag{15}$$

and it has terms containing $\Box^{-2}$. The operator $(-\Box)^2\Pi$ is local but has derivatives of too high an order, so a square root is once again required. Just as in the previous case we obtain a one-parameter set, then introduce the changes $h_{\mu\nu} \to h_{\mu\nu} + \beta\eta_{\mu\nu} h^\lambda{}_\lambda$, and restrict ourselves to the Lagrangian-type equations to arrive at the common Fierz–Pauli set of equations:

$$\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \frac{1+\alpha}{1+2\alpha}(\eta_{\mu\nu}\partial^\lambda \partial^\rho h_{\lambda\rho} + \partial_\mu \partial_\nu h^\lambda{}_\lambda) - \frac{2+4\alpha+3\alpha^2}{2(1+2\alpha)^2}\eta_{\mu\nu}\Box h^\lambda{}_\lambda - \frac{1+\alpha+\alpha^2}{(1+2\alpha)^2} m^2 \eta_{\mu\nu} h^\lambda{}_\lambda + m^2 h_{\mu\nu} = 0. \tag{16}$$

Finally, we consider the massless case. The massless equations are obtained from the massive ones simply by putting $m = 0$. What about the irreducibility conditions now? Note that the massless equations do not lead to corollaries of type (5) and (6). Moreover, the character of the representations at $m = 0$ changes substantially and the conditions (6) now lose their sense. They are replaced by one or more gauge invariances of the equation which make the superfluous degrees of freedom entirely arbitrary, i.e. inessential. For instance, the Proca equation (8) becomes invariant at $m = 0$ under the gauge transformation:

$$a_\mu(x) \to a_\mu(x) + \partial_\mu \phi(x)$$

where $\phi(x)$ is an arbitrary scalar function. The Rarita–Schwinger equations (14) are invariant at $m = 0$ under substitutions

$$\psi_\mu(x) \to \psi_\mu(x) + \partial_\mu \lambda(x) + \frac{\alpha - 1}{2\alpha}\not\partial \gamma_\mu \lambda(x)$$

with arbitrary spinor function $\lambda(x)$, etc.

Now we can transfer all these considerations to the SF case. Note one peculiarity only: each SF contains bosons as well as fermions. To establish the right order of the SF equation operator the following arguments are used. The SF equations must follow from the action principle (Ogievetsky and Mezincescu 1975)

$$S = \int d^4x \, d^4\theta \, \mathscr{L}(x, \theta). \tag{17}$$

Here $\int d^4\theta$ is understood as a Grassman integral (Berezin 1966), i.e. $\int \theta_\alpha \, d\theta^\beta = \delta_\alpha{}^\beta$.

This means that the dimensionality $[d\theta]=-\frac{1}{2}$ (in cm) because $[\theta]=\frac{1}{2}$. Then $[\mathscr{L}]=-2$ since $[S]=0$ (in units $\hbar=c=1$). Let us write the kinetic term in the Lagrangian in the form

$$\mathscr{L}_{\mathrm{K}}=\Phi_i\pi_{ij}\Phi_j$$

where π_{ij} is the equation of motion operator. Now it is clear that

$$[\pi]=-2-2[\Phi]. \tag{18}$$

Finally, the dimensionality of the SF is determined by the dimensionality of the component field with the leading spin. To make this statement clear consider the general scalar SF $\Phi(x,\theta)$. Suppose we are interested in the highest superspin $Y=\frac{1}{2}$ which includes the leading spin-1. This spin is carried by the field $A_\mu(x)$ (see the decomposition (2)) and it is natural to ascribe to it the canonical dimensionality cm^{-1}. Therefore the SF dimensionality equals zero and according to equation (18) $[\pi]=-2$. This is just the dimensionality of the localised superspin-$\frac{1}{2}$ projection operator (Sokatchev 1975)

$$\pi=(-\Box)\Pi^{1/2},\qquad \Pi^{1/2}=1+\frac{1}{4\Box}(\bar{\mathscr{D}}\mathscr{D})^2.$$

Thus we find the equation (Wess 1976)

$$(\Box+\tfrac{1}{4}(\bar{\mathscr{D}}\mathscr{D})^2)\Phi+m^2\Phi=0. \tag{19}$$

As was to be expected, the irreducibility condition $\bar{\mathscr{D}}\mathscr{D}\Phi=0$ follows from equation (19) at $m\neq 0$ and at $m=0$ there arises a gauge invariance

$$\Phi\to\Phi+\bar{\mathscr{D}}\mathscr{D}\Lambda$$

where $\Lambda(x,\theta)$ is an arbitrary scalar superfunction. If equation (19) is written in terms of the component fields and the auxiliary fields are eliminated, a set of the standard equations for a vector, a scalar and two spinors (at $m=0$—for a vector and a spinor) is obtained.

Now we turn to the spinor SF.

4. The spinor superfield

The spinor SF is defined by its decomposition

$$\begin{aligned}\Psi_\alpha(x,\theta)=&\,\psi_\alpha{}^{(1)}(x)+\bar{\theta}^\beta\psi_{\beta\alpha}{}^{(2)}(x)+\tfrac{1}{4}\bar{\theta}\theta\psi_\alpha{}^{(3)}(x)+\tfrac{1}{4}\bar{\theta}\gamma_5\theta\psi_\alpha{}^{(4)}(x)+\tfrac{1}{4}\bar{\theta}\mathrm{i}\gamma^\mu\gamma_5\theta\psi_{\alpha\mu}{}^{(5)}(x)\\&+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}^\beta\psi_{\beta\alpha}{}^{(6)}(x)+\tfrac{1}{32}(\bar{\theta}\theta)^2\psi_\alpha{}^{(7)}(x);\\ \psi_{\beta\alpha}{}^{(2)}=&\,(U_1 1+U_2\gamma_5+\mathrm{i}U_3{}^\mu\gamma_\mu+\mathrm{i}U_4{}^\mu\gamma_\mu\gamma_5+\mathrm{i}U_5{}^{\mu\nu}\sigma_{\mu\nu})_{\beta\alpha}\\ \psi_{\beta\alpha}{}^{(6)}=&\,(u_1 1+u_2\gamma_5+\mathrm{i}u_3{}^\mu\gamma_\mu+\mathrm{i}u_4{}^\mu\gamma_\mu\gamma_5+\mathrm{i}u_5{}^{\mu\nu}\sigma_{\mu\nu})_{\beta\alpha}.\end{aligned} \tag{20}$$

If Ψ_α is a Majorana SF then all the Fermi fields:

$$\psi_\alpha{}^{(1)}(x),\qquad \psi_\alpha{}^{(3)}(x),\qquad \psi_\alpha{}^{(4)}(x),\qquad \psi_{\alpha\mu}{}^{(5)}(x),\qquad \psi_\alpha{}^{(7)}(x)$$

are also Majorana fields and all the Bose fields $U_{1\ldots5}$, $u_{1\ldots5}$ are real.

These fields involve a considerable number of spins, among which the leading spin-$\frac{3}{2}$ is the most interesting (it is connected with the field $\psi_{\alpha\mu}{}^{(5)}$). The leading spin enters into the supermultiplet with the highest superspin-1, which is singled out by the projection operator (Sokatchev 1975)

$$(\Pi^1)_\alpha{}^\beta = \frac{3}{4}\left(1 + \frac{(\bar{\mathscr{D}}\mathscr{D})^2}{4\Box}\right)1_\alpha{}^\beta + \frac{1}{8\Box}\mathrm{i}\partial_\mu \bar{\mathscr{D}}\mathrm{i}\gamma_\nu\gamma_5\mathscr{D}(\sigma^{\mu\nu}\gamma_5)_\alpha{}^\beta \tag{21}$$

or equivalently by the supplementary conditions

$$\bar{\mathscr{D}}\mathscr{D}\Psi_\alpha = 0, \qquad \bar{\mathscr{D}}^\alpha\Psi_\alpha = 0. \tag{22}$$

We wish to find an equation which describes only this superspin-1 but not superspin-$\frac{1}{2}$ and superspin-0.

Let the spin-vector field $\psi_{\alpha\mu}{}^{(5)}$ have the canonical dimensionality $\mathrm{cm}^{-3/2}$. Then the dimensionality of the SF $\Psi_\alpha(x, \theta)$ is $\mathrm{cm}^{-1/2}$ and according to equation (18) the equation operator must have dimensionality cm^{-1}. The localised projection operator (21) has dimensionality cm^{-2} and therefore we have to take the square root of this. There exists a family of such roots with arbitrary parameters ξ, η:

$$\pi(\xi, \nu) = \tfrac{1}{8}\{(\cos\xi + \gamma_5 \sin\xi)(6\mathrm{i}\not{\partial} - \mathrm{i}\gamma_\mu\gamma_5\bar{\mathscr{D}}\mathrm{i}\gamma^\mu\gamma_5\mathscr{D}) + (\cos\eta)(\bar{\mathscr{D}}\mathscr{D} + 3\gamma_5\bar{\mathscr{D}}\gamma_5\mathscr{D}) + (\sin\eta)[\gamma_5(3\bar{\mathscr{D}}\mathscr{D} + \gamma_5\bar{\mathscr{D}}\gamma_5\mathscr{D})]\}.$$

However, all these roots are in fact equivalent because the equations following from them are connected with each other by γ_5 transformations:

$$\Psi_\alpha \to (\mathrm{e}^{\xi\gamma_5}\Psi)_\alpha, \qquad \theta_\alpha \to (\mathrm{e}^{\eta\gamma_5}\theta)_\alpha, \qquad \Psi'(x, \theta') = \mathrm{e}^{\xi\gamma_5}\Psi(x, \theta).$$

Therefore we choose one of these roots ($\xi = \eta = 0$) and write down the equation

$$\tfrac{1}{8}(6\mathrm{i}\not{\partial} + \bar{\mathscr{D}}\mathscr{D} + 3\gamma_5\bar{\mathscr{D}}\gamma_5\mathscr{D} - \mathrm{i}\gamma_\mu\gamma_5\bar{\mathscr{D}}\mathrm{i}\gamma^\mu\gamma_5\mathscr{D})\Psi - m\Psi = 0. \tag{23}$$

Using the fact that equation (23) contains $\pi = \sqrt{(-\Box)}\Pi$ the inverse operator is easily found:

$$\frac{1}{\pi - m} = -\frac{\pi + m}{\Box + m^2}\left(1 + \frac{\Box}{m^2}(1 - \Pi)\right)$$

which defines the Green function and is needed for the perturbation calculations.

Equation (23) can be obtained from the action principle

$$S = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \mathscr{L}(x, \theta) = \frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \bar{\Psi}(\pi - m)\Psi.$$

One can represent the Lagrange density in a more convenient form

$$\mathscr{L} = \tfrac{1}{32}[\bar{\Psi}\mathrm{i}\not{\partial}\Psi - \tfrac{1}{2}(\bar{\mathscr{D}}\gamma_\mu\Psi)^2 + \tfrac{1}{12}(\bar{\mathscr{D}}\sigma_{\mu\nu}\Psi)^2] - \tfrac{1}{2}m\bar{\Psi}\Psi \tag{24}$$

using the algebraic properties of the spinor derivatives $\mathscr{D}_\alpha$ (Salam and Strathdee 1975) and integrating by parts.

To be convinced once more that this Lagrangian describes the superspin-1 multiplet it is useful to write it down in terms of component fields (see equation (20)). The final result will be in its most compact and illustrative form if the superfluous degrees of freedom are excluded, which is usually done by means of the equations of motion. We prefer another more legitimate procedure which involves suitable changes of the field variables (which are apropos also suggested by the equations of motion). After these changes the superfluous fields still remain in the Lagrangian but it becomes evident that they are inessential (as the equations of motion for them are trivial).

So the fermion component fields are replaced by

$$\begin{aligned}
\psi^{(1)} &= \psi \\
\psi^{(3)} &= -\tfrac{1}{4}\phi + \tfrac{1}{4}\gamma_5\chi + \mathrm{i}\gamma_5\gamma_\mu\psi^\mu - \tfrac{1}{3}(\mathrm{i}\not\partial - m)\psi \\
\psi^{(4)} &= \tfrac{3}{4}\chi - \tfrac{1}{4}\gamma_5\phi + \gamma_5(\mathrm{i}\not\partial - m)\psi \\
\psi^{(5)}_\mu &= \psi_\mu + \tfrac{1}{3}\mathrm{i}\gamma_5(\mathrm{i}\partial_\mu - m\gamma_\mu)\psi + \tfrac{1}{4}\mathrm{i}\gamma_5\gamma_\mu\phi - \tfrac{1}{4}\mathrm{i}\gamma_\mu\chi \\
\psi^{(7)} &= \lambda - \tfrac{1}{2}\mathrm{i}\not\partial(\phi + \gamma_5\chi) + \tfrac{2}{3}\gamma_5(\not\partial\gamma_\mu - \partial_\mu)\psi^\mu - \Box\psi + \tfrac{2}{3}m(\mathrm{i}\not\partial - 4m)\psi + m\gamma_5\chi
\end{aligned} \tag{25a}$$

and the boson component fields by

$$\begin{aligned}
&U_1 \equiv A, \qquad U_2 \equiv B, \qquad U_{3\mu} \equiv V_\mu \\
&U_{4\mu} = A_\mu - \frac{1}{m}\epsilon_{\mu\lambda\sigma\kappa}\partial^\lambda E^{\sigma\kappa} \\
&U_{5\mu\nu} = E_{\mu\nu} + \frac{1}{2m}(\partial_\mu V_\nu - \partial_\nu V_\mu) \\
&u_1 = a + \partial_\mu V^\mu \\
&u_2 = b - \partial_\mu A^\mu - mB \\
&u_{3\mu} = v_\mu + \partial_\mu A + 2\partial^\nu E_{\nu\mu} + 2mV_\mu + \frac{1}{m}(\Box V_\mu - \partial_\mu\partial^\nu V_\nu) \\
&u_{4\mu} = a_\mu - \partial_\mu B + \epsilon_{\mu\lambda\sigma\kappa}\partial^\lambda E^{\sigma\kappa} \\
&u_{5\mu\nu} = e_{\mu\nu} + \tfrac{1}{2}(\partial_\mu V_\nu - \partial_\nu V_\mu) + \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda A^\rho - 2mE_{\mu\nu} - \frac{1}{m}(\Box E_{\mu\nu} + \partial_\mu\,\partial^\lambda E_{\nu\lambda} - \partial_\nu\,\partial^\lambda E_{\mu\lambda}).
\end{aligned} \tag{25b}$$

Then after integration over $\mathrm{d}^4\theta$ the Lagrangian for the fields takes the form

$$\begin{aligned}
\mathscr{L}(x) = {}& \bar\psi^\mu \mathrm{i}\not\partial\psi_\mu - 2\psi^\mu \mathrm{i}\partial_\mu\gamma^\nu\psi_\nu + \bar\psi^\mu\gamma_\mu \mathrm{i}\not\partial\gamma^\nu\psi_\nu + m\bar\psi^\mu\gamma_\mu\gamma^\nu\psi_\nu \\
& - m\bar\psi^\mu\psi_\mu - \tfrac{4}{3}m^2(\bar\psi \mathrm{i}\not\partial\psi - m\bar\psi\psi) + \tfrac{3}{4}\bar\lambda\gamma_5\chi + \tfrac{1}{4}m\bar\phi\phi \\
& + 8(V^\mu\Box V_\mu - V^\mu\partial_\mu\partial^\nu V_\nu) + 8m^2V_\mu^2 - 8E^{\mu\nu}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda\partial_\kappa\epsilon^{\rho\kappa\alpha\beta}E_{\alpha\beta} \\
& - 16m^2E_{\mu\nu}^2 + 4e_{\mu\nu}^2 - 2v_\mu^2 + 8maA - 8ma^\mu A_\mu - 4b^2 + 4m^2B^2.
\end{aligned} \tag{26}$$

The adequacy of the choice of the SF equation is confirmed. Really the irreducible superspin-1 representation contains spin-$\frac{3}{2}$ (spin-vector field ψ_μ having the standard Rarita–Schwinger Lagrangian), two spins-1 (vector V_μ and antisymmetric tensor $E_{\mu\nu}$ fields with correct Lagrangians) and spin-$\frac{1}{2}$ (the field $m\psi$). All the other fields are evidently inessential. Furthermore it is easy to verify that fields ψ_μ, V_μ, $E_{\mu\nu}$ and $m\psi$ form an invariant subspace under the supersymmetry transformations.

Now we are going to discuss the zero mass case. The corresponding Lagrangian is obtained by setting $m = 0$ in equation (24). However we cannot set $m = 0$ in equation (26) because this form is obtained from equation (24) by the singular field changes (25) at $m \to 0$. This can be explained as follows. The original decomposition (20) contains the spinor field ψ with dimensionality $\mathrm{cm}^{-1/2}$. Multiplying it by the mass we obtain the canonical dimensionality $\mathrm{cm}^{-3/2}$. The corresponding part in the Lagrangian (26) is proportional to m^2 which ensures that it vanishes as $m \to 0$. The situation with the boson fields is different. In the decomposition (20) there are no fields having spin-1 and dimensionality less than the canonical one of cm^{-1} which would enter into the Lagrangian (26) with a multiplier proportional to m and vanish as $m \to 0$. Therefore at $m = 0$ in the field changes (25) all the terms which contain m must be omitted from both the denominators and the numerators. Then the Lagrangian may be written

$$\mathscr{L}(x) = \bar{\psi}^\mu \mathrm{i}\partial\!\!\!/ \psi_\mu - 2\psi^\mu \mathrm{i}\partial_\mu \gamma^\nu \psi_\nu + \bar{\psi}^\mu \gamma_\mu \mathrm{i}\partial\!\!\!/ \gamma^\nu \psi_\nu + \tfrac{3}{4}\bar{\lambda}\gamma_5\chi$$
$$+16(V^\mu \Box V_\mu - V^\mu \partial_\mu \partial^\nu V_\nu) - 8E^{\mu\nu}\epsilon_{\mu\nu\lambda\rho}\partial^\lambda a^\rho + 4e_{\mu\nu}^2 - 2v_\mu^2 - 4b^2. \qquad (27)$$

The essential fields here are ψ_μ (chiralities $\pm\frac{3}{2}$) and V_μ (chiralities ± 1). (Recall that the zero mass supermultiplets include only two successive chiralities.) The equations for the other fields are trivial. In particular

$$\partial^\lambda \epsilon_{\rho\lambda\mu\nu} E^{\mu\nu} = 0, \qquad \epsilon_{\mu\nu\lambda\rho}\partial^\lambda a^\rho = 0$$

and we have

$$E_{\mu\nu} = \partial_\mu w_\nu - \partial_\nu w_\mu, \qquad a_\rho = \partial_\rho s \qquad (28)$$

where w_μ and s are arbitrary vector and scalar fields, respectively. In other words the fields $E_{\mu\nu}$ and a_ρ are not essential due to the invariance of the Lagrangian (27) under the gauge transformations

$$E_{\mu\nu} \to E_{\mu\nu} + \partial_\mu w_\nu - \partial_\nu w_\mu, \qquad a_\rho \to a_\rho + \partial_\rho s.$$

The Lagrangian (27) is also invariant under the standard gauge transformations of the vector and spin-vector fields

$$V_\mu \to V_\mu + \partial_\mu f, \qquad (29a)$$

$$\psi_\mu \to \psi_\mu + \partial_\mu \xi, \qquad (29b)$$

where f is the arbitrary scalar function and ξ is the arbitrary spinor function. All these transformations have a SF form. Indeed, the Lagrangian (24) allows the gauge transformations (at $m = 0$)

$$\Psi_\alpha \to \Psi_\alpha + \mathscr{D}_\alpha \Lambda \qquad (30a)$$

$$\Psi_\alpha \to \Psi_\alpha + (\mathrm{i}\partial\!\!\!/ \gamma_5 \mathscr{D})_\alpha \Sigma \qquad (30b)$$

where $\Lambda(x, \theta)$ and $\Sigma(x, \theta)$ are arbitrary scalar superfunctions. The first, equation (30*a*), is connected with the invariance (29*a*) of the Proca equation and it enables us to construct the generalisation of the Yang–Mills theory mentioned in §1. The second transformation (30*b*) provides the gauge freedom (29*b*) of the Rarita–Schwinger equation and causes some trouble when the interaction is introduced.

Ending this section we wish to stress once more the compactness and effectiveness of the SF formalism in comparison with the treatment of the supersymmetric models in terms of the component fields.

Unfortunately we are not yet familiar enough with the SF language. Because of this we often need lengthy and tiresome calculations in terms of the field components to achieve greater confidence and apparent clarity.

In conclusion we note that the equations of motion for other SF can be obtained in a similar way. In connection with supergravity we are especially interested in the vector SF, the Lagrange theory of which will be discussed in a separate paper.

References

Aurilia A and Umezawa H 1969 *Phys. Rev.* **182** 1682
Belinicher V 1974 *Theor. Math. Phys.* **20** 320
Berezin F 1966 *The Method of Second Quantization* (New York, London: Academic Press)
Ferrara S, Wess J and Zumino B 1974 *Phys. Lett.* **51B** 239
Ferrara S and Zumino B 1975 *Nucl. Phys.* B **87** 207
Lichtman E 1971 *Lebedev Institute of Physics Preprint* No. 41
Ogievetsky V and Mezincescu L 1975 *Usp. Fiz. Nauk* **117** 637
Ogievetsky V and Sokatchev E 1976 (*JETP Lett.* **23** 58) *Z. Eksp. Teor. Fiz. Pis. Red.*
Salam A and Strathdee J 1974 *Nucl. Phys.* B **80** 499
—— 1975 *Phys. Rev.* D **11** 1521
Sokatchev E 1975 *Nucl. Phys.* B **99** 96
Wess J 1976 *Current Problems in Elementary Particle and Mathematical Physics* (New York: Wien) p 475